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Coordinate noncommutativity as low energy effective dynamics

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Abstract

Coordinate noncommutativity, rather than being introduced through deformations of operator products, is achieved by coupling an auxiliary system with large energy excitations to the one of interest. Integrating out the auxiliary dynamics, or equivalently taking ground state expectation values, leads to the desired coordinate noncommutativity. The product responsible for this noncommutativity is different from the Groenewold-Moyal one. For products of operators at unequal times, this procedure differs from the normal, commutative one, for time differences smaller than ones characterized by the auxiliary system; for larger times the operator algebra reverts to the usual one.

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Noncommutativity between space coordinates is most commonly introduced by way of the Groenewold-Moyal [1] star product deformation of the ordinary product; it suffices to define this for exponential functions as the product of other functions is obtained from Fourier transforms,

$$\exp i\mathbf{k}\cdot\mathbf{r}\star\exp i\mathbf{q}\cdot\mathbf{r}=\exp i\left[(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}\right]\exp\left(\frac{i}{2}\theta_{ab}k_a q_b\right)\ . \tag{1}$$

As all vectors are two dimensional, we may set $\theta_{ab}=\theta\epsilon_{ab}$, where θ determines the magnitude of the noncommutativity. The above is the unique *associative* deformation [2] of the ordinary product. Such noncommutative geometries arise on D-branes in magnetic backgrounds [3]. Noncommutativity induced by magnetic fields can be applied to the quantum mechanics of point particles [4, 5, 6]. In the latter case it is noted that in the dynamics of a charged particle restricted to the lowest Landau level of a uniform magnetic field the momenta decouple and one of the space coordinates, x, becomes the canonical momentum of the other one, y; the resultant commutator of x and y is inversely proportional to the applied field.

An advantage of this second approach is that an ordinary product algebra is maintained. However, the disappearance of the momenta from the dynamics of a charged system in a strong magnetic field prevents us from applying this method to problems where we wish to keep both momenta and coordinates, with commutation relations $[p_a, p_b]=0$ and $[p_a, r_b]=-i\delta_{ab}$ ($\hbar$ is set to 1) while the space-space commutator becomes

$$[r_a, r_b]=-i\theta\epsilon_{ab}\ . \tag{2}$$

In this work we achieve this goal by correlating the dynamics of the system of interest to an auxiliary one and then integrating out the second system. The resultant effective theory, valid for energies lower than the excitation energies of the auxiliary system, yields the desired non trivial space-space commutator. The price paid is that the effective low energy theory is nonlocal in time. It is, of course, possible to forgo the auxiliary system and postulate the minimum effective theory as needed to produce noncommutativity. We begin with this approach and then motivate it and make it more precise by introducing the auxiliary system leading to the modified dynamics as a low energy effective theory. The minimal effective theory reproduces the Groenewold-Moyal product. The product resulting from the more fundamental approach is significantly different; the commutators in the two approaches are identical.

Starting with the action for a problem with normal commuting coordinates,

$$S_0 = \int dt \left[\mathbf{p} \cdot \dot{\mathbf{r}} - H_0(\mathbf{p}, \mathbf{r}) \right] \,, \tag{3}$$

the commutation relation (2) can be implemented by adding to S_0 a term, nonlocal in time,

$$S_{\rm nc} = \frac{\epsilon_{ab}}{4} \int dt\, dt' \dot{p}_a(t) \epsilon(t-t') \Theta(|t-t'|) \dot{p}_b(t') \,; \tag{4}$$

where $\Theta(|t|)$ is a function that goes to zero for $|t|$ much larger than some characteristic time $\tau = 1/\omega$ and $\Theta(0) = \theta$. τ is taken to be much smaller than any time relevant to H_0; we assume that we are dealing with momenta p and potentials $V(x,y)$ that satisfy

$$\tau \frac{\mathbf{p}}{m} \cdot \boldsymbol{\nabla} V(x,y) << V(x,y) \,; \tag{5}$$

namely for space intervals traversed in times less than τ, the potential dos not vary much.

Expectation values of products of position operators are obtained from the generating functional $Z[\mathbf{j}(t)]$ given by the path integral

$$Z[\mathbf{j}(t)] = \int [d\mathbf{p}][d\mathbf{r}] \exp i \left[S_0 + S_{\rm nc} + \int dt \mathbf{j}(t) \cdot \mathbf{r}(t) \right] \,; \tag{6}$$

the time ordered product of the $r_a(t)$'s is

$$\langle T \left[r_a(t_a) r_b(t_b) \cdots \right] \rangle = \frac{1}{Z[0]} \frac{-i\delta}{\delta j_a(t_a)} \frac{-i\delta}{\delta j_b(t_b)} \cdots Z[\mathbf{j}(t)] \,, \tag{7}$$

and the expectation value is taken in the joint ground of $H + H_{\rm nc}$. Condition (5) permits an explicit evaluation of the modification to (6) due to $S_{\rm nc}$

$$Z[\mathbf{j}(t)] = \exp i \left[\frac{\epsilon_{ab}}{4} \int dt\, dt' j_a(t) \epsilon(t-t') \Theta(|t-t'|) j_b(t') + \text{usual terms} \right] \,. \tag{8}$$

The usual terms do not yield in any coordinate noncommutativity; the term due to $S_{\rm nc}$ results in

$$\begin{aligned} \ldots x(t+\epsilon) y(t) \ldots &= \ldots \frac{-i}{2} \Theta(\epsilon) \ldots \,; \\ \ldots y(t+\epsilon) x(t) \ldots &= \ldots \frac{+i}{2} \Theta(\epsilon) \ldots \,, \end{aligned} \tag{9}$$

with the dots indicating other operators at times later and earlier than t. The limit $\epsilon \to 0$ this yields the desired commutation relation. (It is interesting to note that in the path integral formulation the usual momentum-coordinate commutation relations also arise [7]

by considering time ordered products of $p(t) = m\dot{x}(t)$ and $x(t')$ and letting $t' \to t$.) Similarly we find

$$\lim_{\epsilon \to 0} \langle \exp i\mathbf{k} \cdot \mathbf{r}(t+\epsilon) \exp i\mathbf{q} \cdot \mathbf{r}(t) \rangle = \langle \exp i(\mathbf{k}+\mathbf{q}) \cdot \mathbf{r}(t) \rangle \exp \frac{i\theta}{2} \epsilon_{ab} k_a q_b \,, \tag{10}$$

as stated earlier, the usual Groenewold-Moyal product.

Can the action $S_{\rm nc}$ be obtained from dynamics local in time? To this end we will couple the system of interest, H_0, to a one dimensional auxiliary one with a momentum operator ρ_x and a position operator ρ_y with

$$[\rho_x, \rho_y] = -i \,; \tag{11}$$

it is for later convenience that we denote the momentum conjugate to ρ_y by ρ_x. For simplicity the Hamiltonian for this auxiliary system is taken as

$$H_{\rm aux} = \omega(\rho_x^2 + \rho_y^2)/2 \,; \tag{12}$$

ω will be chosen larger than the energies of interest in H_0. The coupling is accomplished by replacing the operators $\mathbf{r}$ by $\mathbf{R} = (X, Y)$ with

$$\begin{aligned} X &= x + \sqrt{\theta}\rho_x \\ Y &= y + \sqrt{\theta}\rho_y \,. \end{aligned} \tag{13}$$

This coupling is unusual in that while the position operator y displaced by a term proportional to ρ_y, the other position operator x is displaced by a term proportional to the conjugate momentum ρ_x. Further on we return to this point. For large ω the excitations of $H_{\rm aux}$ will be limited insuring that $\langle \rho_x^2 \rangle$ and $\langle \rho_y^2 \rangle$ are of order one and in $\langle (\mathbf{R}-\mathbf{r})^2 \rangle \sim \theta$. In addition to the replacement of $\mathbf{r}$ by $\mathbf{R}$ in H_0, the source term also changes to $\mathbf{j}(t) \cdot \mathbf{R}(t)$. It is clear that $[X, Y] = -i\theta$. The analog of (6) is

$$Z[\mathbf{j}(t)] = \int [d\mathbf{p}][d\mathbf{r}][d\boldsymbol{\rho}] \exp i \int dt \left[\mathbf{p} \cdot \dot{\mathbf{r}} + \rho_x \dot{\rho}_y - H(\mathbf{p}, \mathbf{R}) - H_{\rm aux}(\boldsymbol{\rho}) + \mathbf{j}(t) \cdot \mathbf{R}(t)\right] . \tag{14}$$

After changing the variables of integration from $\mathbf{r}$ to $\mathbf{R}$, the integration over ρ_x and ρ_y maybe performed yielding

$$Z[\mathbf{j}(t)] = \int [d\mathbf{p}][d\mathbf{R}] \exp i \int dt \left[\mathbf{p} \cdot \dot{\mathbf{R}} - H(\mathbf{p}, \mathbf{R}) + \mathbf{j}(t) \cdot \mathbf{R}(t) + S_{\rm nc}\right] , \tag{15}$$

with

$$S_{\rm nc} = \frac{\epsilon_{ab}}{4}\theta \int dt\, dt' \dot{p}_a(t) \epsilon(t-t') e^{-i\omega|t-t'|} \dot{p}_b(t') + \frac{i\theta}{4} \int dt\, dt' \dot{\mathbf{p}}(t)\cdot\dot{\mathbf{p}}(t') e^{-i\omega|t-t'|} \,. \tag{16}$$

The first term above is of the form proposed in (4) with $\Theta(|t|) = \theta \exp(-i\omega t)$. While the second term does not contributing to coordinate noncommutativity it has the interesting consequence of making the resulting prescription for the product of operators different from the star product one. The noncommutative product (10) becomes

$$\begin{aligned} \lim_{\epsilon\to 0} \exp\left[i\mathbf{k}\cdot\mathbf{r}(t+\epsilon)\right] \;\times\; & \exp\left[i\mathbf{q}\cdot\mathbf{r}(t)\right] \\ = \;& \exp\left[i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}(t)\right] \exp\left(\frac{i\theta}{2}\epsilon_{ab}k_a q_b\right) \exp\left[-\frac{\theta}{4}(\mathbf{k}+\mathbf{q})^2\right] \,. \end{aligned} \tag{17}$$

The extra term, compared to (10), has its origin in the second term of (16) and may be understood by looking at an operator formulation of this procedure. Position dependent operators, $O(\mathbf{r}) = \int d\mathbf{q}\, \tilde{O}(\mathbf{q}) \exp \mathbf{iq}\cdot\mathbf{r}$, are replaced by (see (13))

$$O(\mathbf{r}) \to O(\mathbf{R}) = \int d\mathbf{q}\, \tilde{O}(\mathbf{q}) \exp\left[\mathbf{iq}\cdot(\mathbf{r}+\sqrt{\theta}\boldsymbol{\rho})\right] \,. \tag{18}$$

The properties of products is determined by studying products of $\exp i\left[\mathbf{q}\cdot(\mathbf{r}+\sqrt{\theta}\boldsymbol{\rho})\right]$'s;

$$\begin{aligned} \exp\left[i\mathbf{k}\cdot(\mathbf{r}+\sqrt{\theta}\boldsymbol{\rho})\right] \;\times\; & \exp\left[i\mathbf{q}\cdot(\mathbf{r}+\sqrt{\theta}\boldsymbol{\rho})\right] \\ = \;& \exp\left[i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}\right] \exp(i\theta\epsilon_{ab}k_a q_b) \exp\left[i\sqrt{\theta}(\mathbf{k}+\mathbf{q})\cdot\boldsymbol{\rho}\right] \,. \end{aligned} \tag{19}$$

Taking the $H_{\rm aux}$ vacuum expectation value of the above, equivalent to integrating out the ρ's in (15), agrees with (17).

It is straightforward to extend (19) to more than two factors.

$$\prod_n \exp\left[i\mathbf{k}_n\cdot(\mathbf{r}+\sqrt{\theta}\boldsymbol{\rho})\right] = \exp\left[i\sum_n \mathbf{k}_n\cdot\mathbf{r}\right] \exp\left[i\theta\epsilon_{ab}\sum_{n<m} k_{na}k_{mb}\right] \exp\left[i\sqrt{\theta}\sum_n \mathbf{k}_n\cdot\boldsymbol{\rho}\right] \,. \tag{20}$$

Although this product, before any expectation values are taken, is associative, it ceases to be once the $H_{\rm aux}$ ground state matrix elements are taken. Due to this non associativity, uniqueness theorems [2] for deformations do not apply and this deformation *is different* from the Groenewold-Moyal one.

The replacement (13) of $\mathbf{r}$ by $\mathbf{R}$ may appear somewhat unnatural in that the y coordinate is shifted by a a a multiple of the coordinate ρ_y while the coordinate x is shifted by a multiple of the conjugate momentum ρ_x. This may be avoided by extending the auxiliary system to

two dimensions, considering $\boldsymbol{\rho}$ as a coordinate vector, introducing new conjugate momenta $\boldsymbol{\pi}$, with $[\pi_a, \rho_b] = -i\delta_{ab}$, and placing the system in a strong magnetic field. $H_{\rm aux}$ is replaced by

$$H'_{\rm aux} = \frac{\pi_x^2}{2\mu} + \frac{(\pi_y - \rho_x)^2}{2\mu} + \frac{\omega}{2}\rho^2\,; \tag{21}$$

we have scaled the variables to set $eB = 1$. The ω term breaks the degeneracy of the lowest Landau level with energies in this level being $E_n = \left[\sqrt{4\omega\mu+1}(n+1) - n\right]/(2\mu)$. The limit $\mu \to 0$, specifically $\mu << 1/\omega$, forces this system into the lowest Landau level [4, 5, 6]. ρ_x does become the momentum conjugate to ρ_y and the preceding discussion holds. For non zero μ, $\exp i\omega|t-t'|$ in (16) is replaced by $[\exp(i\omega|t-t'|) - \exp(i|t-t'|/\mu)]$. At equal times ρ_x and ρ_y commute and (13) doesn't introduce any equal time coordinate noncommutativity. It is interesting to look at $[x(t), x(t')]$ for $t \neq t'$. We have three regions of $|t-t'|$ to consider. For $|t-t'| < \mu$ the commutator $[x(t), y(t')] \sim 0$; for $\mu < |t-t'| < 1/\omega$ the preceding discussions hold and the commutator $\sim \theta$, while for $|t-t'| > 1/\omega$ the dynamics of H_0 determine the commutator.

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